IISER2009Varsha P111Mechanics

Lect2 One Dimensional Motion

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1 Newton's Law

Newton's second law says that

$$m\frac{d^2q}{dt^2} = F(q, v).$$

The constant m is the mass of the particle: it measures the amount of matter it contains. $q(t)$ is the position on the particle as a function of time t.We only consider the case of a particle that can move along a line. For example, a railway compartment on a long track. $F(q, v)$ is the force acting on the particle when it is at position q with velocity $v = \frac{dq}{dt}$. We assume that the force has no direct dependence on time: the force for a given position and velocity is the same no matter what the time.

The basic problem of mechanics is to find the position as a function of time, given the initial position and velocity.

$$q_0 = q(0), \; v_0 = v(0).$$

2 Free Particle

The simplest case is when the force is zero. In this case the acceleration is zero. The velocity is a constant. The position is a linear function of time.

$$q(t) = q_0 + v_0 t.$$

3 Constant Force

If the force is a constant (independent of position and velocity) the acceleration $a = \frac{F}{m}$ is a constant as well. Then

$$\frac{dv}{dt} = a.$$

Then .the velocity is a linear function

$$v(t) = v_0 + at.$$

The position satisfies

$$\frac{dq}{dt} = v_0 + at$$

so that

$$q(t) = q_0 + v_0 t + \frac{1}{2}at^2.$$

The acceleration due to gravity at the surface of the Earth is roughly independent of height (as long the height is a few kilometers or less). For all bodies this is about 9.8 ms^{-2}. It is pointed downward.

4 The Simple Harmonic Oscillator

The next case is when the force is proportional to distance from some fixed point (the origin). It can either point away from the origin or towards the origin. We consider the case where it points to the origin always.

$$F = -kq, \ k > 0.$$

We have then

$$\frac{dq}{dt} = v$$

$$\frac{dv}{dt} = -\frac{k}{m}q.$$

This is called the Simple Harmonic Oscillator. It describes a particle that moves back and forth (oscillates) around the origin. To solve the equation remember the equations of calculus

$$\frac{d\sin\theta}{d\theta} = \cos\theta$$

$$\frac{d\cos\theta}{d\theta} = -\sin\theta$$

So

$$q = A\sin[\omega t + \phi]$$

is a solution for any constants A and ϕif

$$\omega = \sqrt{\frac{k}{m}}.$$

(Prove this!)
The constants $A$and $\phi$are determined by the initial conditions

$$q_0 = A \sin \phi,$$

$$v_0 = A\omega \cos \phi.$$

The position and velocity are periodic funtion.

$$q(t + T) = q(t)$$

if

$$T = \frac{2\pi}{\omega}.$$

(Prove this also.)
You will see this example appear everywhere in physics: it is a most fundamental system.

5 The Inverse Square Law

Imagine a particle (maybe a comet) that falls into the Sun. The Sun is located at the origin. The comet is coming in from the right, so that its initial position is a positive number. The Gravitational force is pointed to the left so it is negative:

$$F = -\frac{GMm}{q^2}.$$

Here m is the mass of the comet and $M$the mass of the Sun. G is Newton's gravitational constant.
We have

$$\frac{dv}{dt} = -\frac{k}{q^2}$$

$$k = GM.$$

This is harder to solve, as it is an example of a non-linear equation. The trick is to find a combination of position and velocity that is constant: a conserved quantity. Now, recall that

$$v = \frac{dq}{dt},$$

$$\frac{d}{dt}\left[\frac{k}{q}\right] = -\frac{k}{q^2}v.$$

Also

$$v\frac{dv}{dt} = \frac{d}{dt}\left[\frac{1}{2}v^2\right].$$

By multiplying the above equation by v,

$$v\frac{dv}{dt} = -\frac{k}{q^2}v$$

$$\frac{d}{dt}\left[\frac{1}{2}v^2\right] = \frac{d}{dt}\left[\frac{k}{q}\right]$$

$$\frac{d}{dt}\left[\frac{1}{2}v^2 - \frac{k}{q}\right] = 0.$$

Thus

$$\frac{1}{2}v^2 - \frac{k}{q} = C$$

a constant.
That is

$$v = -\sqrt{2\left[C + \frac{k}{q}\right]}$$

The sign negative if the particle is falling in. Thus we get

$$\frac{dq}{dt} = -\sqrt{2\left[C + \frac{k}{q}\right]}$$

$$\frac{dq}{\sqrt{C + \frac{k}{q}}} = -\sqrt{2}dt.$$

Complete the solution of this problem. If the particle starts at a position q_0 *at rest, will it reach the origin in a finite time? If so what is that time?*

6 Dissipation

Consider a force that depends on velocity:

$$F(q, v) = -\gamma v, \gamma > 0.$$

The simplest dependence is a linear function. The sign is chosen such that the force is always acting in the direction opposite to velocity: the system always slows down. This is the usual physical situation.

The solution to

$$m\frac{dv}{dt} = -\gamma v$$

is

$$v(t) = v_0 e^{-\frac{\gamma}{m}t}.$$

It will come to a rest eventually.

If there is also a component to the force that is a constant

$$F(q, v) = F_0 - \gamma v$$

$$m\frac{dv}{dt} = -\gamma v + F_0$$

has solution

.

$$v(t) = \frac{F_0}{\gamma} + Ae^{-\frac{\gamma}{m}t}$$

Determine the constant in terms of the initial velocity.

7 Nonlinear Oscillators

Suppose that the force is independent of velocity; and that it vanishes at some point, called the point of equilibrium. If the derivative of the force is negative at the equilibrium is negative, *it is a stable equilibrium.*A small displacement in either direction will be opposed by the force.

$$m\frac{dv}{dt} = F(q)$$

If the displacement from the equilibrium is not too large (i.e., the force has negative derivative within its range of motion) the system will oscillate. The key is to derive a conservation law. As we did for the SHO, we multiply the above equation by v to get

$$mv\frac{dv}{dt} = F(q)v$$

$$\frac{d}{dt}\left[\frac{1}{2}mv^2\right] = -\frac{d}{dt}U(q)$$

where $U(q)$is a function such that

$$F(q) = -\frac{dU}{dq}.$$

It is called the potential. In the case of one dimensional motion such a potential always exists: it is the negative integral of the force. Thus

$$\frac{d}{dt}\left[\frac{1}{2}mv^2 + U(q)\right] = 0$$

The quantity

$$E = \frac{1}{2}mv^2 + U(q)$$

is the total energy of the system: the first term is the kinetic energy and the second is the potential energy. At the equilibrium point, the potential energy has a minimum:

$$\frac{dU}{dq} = 0, \quad \frac{d^2U}{dq^2} > 0.$$

Because the force vanishes; and it has negative derivative. Thus there is a range of values of q in which U is a convex function (has positive second derivative). (For the SHO this range is the whole line.) Our analysis remains valid in this entire range.

We can solve for velocity

$$v = \pm\sqrt{2\frac{E - U(q)}{m}}$$

$$\frac{dq}{dt} = \pm\sqrt{2\frac{E - U(q)}{m}}$$

$$dt = \pm\frac{dq}{\sqrt{2\frac{E-U(q)}{m}}}$$

$$t = \pm\int^q \frac{dq}{\sqrt{2\frac{E-U(q)}{m}}}$$

For most potentials, this integral cannot be evaluated in terms of standard functions. But even without that we can say something about the nature of the motion.

Exercise: For the special case $U(q) = \frac{1}{2}kq^2$ *this integral can be evaluated in terms of the* arcsin *function. Then you can derive the solution* $q(t) = A\sin[\omega t + \phi]$ *by inverting.*

As long as $U(q)$is convex, and energy is greater than the minimum of $U(q)$, there are two points q_1 and $q_2$at which the denominator vanishes. These are called turning points: looking back, these are points at which the velocity vanishes.

$$E = U(q_1) = U(q_2).$$

The system oscillates between the two turning points forever. The motion is periodic. The time to go from one turning point to the other is

$$\int_{q_1}^{q_2} \frac{dq}{\sqrt{2\frac{E-U(q)}{m}}}$$

The period of the system is twice this:

$$T(E) = 2\int_{q_1}^{q_2} \frac{dq}{\sqrt{2\frac{E-U(q)}{m}}}.$$

For the Simple Harmonic Oscillator, this integral just happens to be independent of energy. But for most other oscillators, the period depends on the energy.

8 An Inverse Problem

Given the force we can, as above, find the period of the oscillation as a function of energy. Suppose you know the period as a function of energy $T(E)$.Can you determine the force? This is an example of an *inverse problem* in physics. It turns that there is a way to do this, but it is much harder: you have to solve integral equations rather than differential equations. And there may be more than one solution: different force laws may lead to the same period function. If the Force is assumed to be an odd function

$$F(-q) = -F(q)$$

there is excatly one solution. See the discussion in Landau-Lifshitz Vol. 1 for the method of solution.

9 Dissipation

Dissipative systems cannot be studied as above because the energy is not conserved. Let us consider to be specific a simple model of dissipation (called Ohmic model) where the velocity dependence is linear in velocity.

$$F(q,v) = -\gamma(q)v - F_1(q)$$

In order that the dissipation always oppose the motion of the system we postulate

$$\gamma(q) > 0.$$

$$m\frac{dv}{dt} = -\gamma(q)v - F_1(q)$$

It follows by multiplying by v as above that

$$\frac{d}{dt}\left[\frac{1}{2}mv^2 + U(q)\right] = -\gamma(q)v^2.$$

$$F_1(q) = -\frac{dU}{dq}$$

Thus such a system always loses energy until it comes to a rest:

$$\frac{dE}{dt} < 0.$$

10 Hamilton-Jacobi Theory

A point of view on mechanics somewhat different from Newton's has turned out to be very important. We introduce a quantity (action) S such that

$$p = \frac{dS}{dq}$$

For a conservative system (when energy is a constant)

$$\frac{p^2}{2m} + U(q) = E.$$

This becomes a differential equation for S

$$\frac{1}{2m}\left[\frac{dS}{dq}\right]^2 + U(q) = E.$$

which can be solved by

$$S(q,E) = \int_{q_1}^{q} \sqrt{2m\left[E - U(q)\right]}dq.$$

If you draw a curve of constant energy in the (q,p)-plane (called the phase space), this is the area under it upto the point q. Once we know the action, the momentum is determined as a function of position by differentiation w.r.t. q. The time dependence of position is also given through the formula (more precisely it gives the time as a function of postion, which then we have to invert):

$$t = \frac{\partial S}{\partial E}.$$

(Prove this.)

11 H-J Theory For Dissipative Systems

I couldn't resist the temptation to sneak in some of my own recent research into this introductory course. The case of one dimensional systems is a place to try out ideas about dissipative systems.

If energy is not conserved we have to follow a different strategy. Starting with

$$\frac{d}{dt}\left[\frac{1}{2}mv^2 + U(q)\right] = -\gamma(q)v^2$$

we get

$$\frac{dq}{dt}\frac{d}{dq}\left[\frac{1}{2}mv^2 + U(q)\right] = -\gamma(q)v^2.$$

$$\frac{d}{dq}\left[\frac{1}{2}mv^2 + U(q)\right] = -\gamma(q)v.$$

$$\frac{d}{dq}\left[\frac{1}{2m}p^2 + U(q)\right] = -\gamma(q)\frac{p}{m}.$$

Let

$$dQ = \frac{1}{m}\gamma(q)dq$$

$$Q(q) = \frac{1}{m}\int^{q}\gamma(q)dq.$$

Define a new function $V(Q)$ such that

$$U(q) = V(Q(q)).$$

Then

$$\frac{d}{dQ}\left[\frac{1}{2m}p^2 + V(Q)\right] = -p.$$

Now put

$$p = \frac{dS}{dQ}$$

to get

$$\frac{d}{dQ}\left[\frac{1}{2m}\left(\frac{dS}{dQ}\right)^2 + V(Q) + S\right] = 0.$$

Thus even for dissipative systems we get a kind of conservation law

$$\frac{1}{2m}\left(\frac{dS}{dQ}\right)^2 + V(Q) + S = \epsilon.$$

This is also a Hamilton-Jacobi equation, but the presence of Sitself in addition to its derivative makes it diffiult to solve it in general.

12 The Pendulum

In physics we look for simple systems that also describe something that appears in nature. It is not hard to find simple things that don't appear anywhere in nature: we can always make up pretty stuff but we call that mathematics. Also, most things in nature are too hard to understand in detail physically: a bird or a river are just too complicated. One of the early successes of physics was understanding the pendulum: a heavy ball suspended from some point. To make it even simple we only allow it to move in one plane: not rotate around in the horizontal direction. This is a mechanical system with one degree of freedom: if you know the angle that the line connecting the ball to its point of suspension makes with the vertical, you know its position. This angle θ is the co-ordinate of the problem.

In reality the pendulum loses some energy each time it moves. But let us ignore this: the point of suspension is well-oiled and the resistance due to air is a small effect for a heavy enough ball. Then the kinetic plus potential energy of the ball is a constant.

Let the distance of the ball from from its point of suspension be l, constant. If $\theta = 0$,the ball is at the bottom of its swing and the potential energy is a minimum. A little geometry will let you find the height of the ball above this point as a function of θ :

$$l(1-\cos\theta).$$

Prove this!

Thus the potential energy is

$$U(\theta) = mgl(1-\cos\theta).$$

The kinetic energy can be found if you know its velocity. The ball moves along an arc of a circle centered at the point of suspension and of radius l. In a small time $dt$the angle changes by $d\theta$ and the distance it moves is $ld\theta$.So the velocity is $l\frac{d\theta}{dt}$. So the kinetic energy is

$$\frac{1}{2}ml^2\left[\frac{d\theta}{dt}\right]^2.$$

Since the energy is conserved

$$\frac{1}{2}ml^2\left[\frac{d\theta}{dt}\right]^2 + mgl(1-\cos\theta) = E$$

We can rewrite this as

$$1-\cos\theta = 2\sin^2\frac{\theta}{2}.$$

If we put $\phi = \frac{\theta}{2}$

$$2ml^2\left[\frac{d\phi}{dt}\right]^2 + 2mgl\sin^2\phi = E$$

For small oscillations we can approximate $\sin\phi \approx \phi$and this becomes a simple harmonic oscillator of period $T = 2\pi\sqrt{\frac{l}{g}}$. *(Prove this.)*

More generally,

$$\left[\frac{d\phi}{dt}\right]^2 = \frac{E}{2ml^2} - \frac{g}{l}\sin^2\phi.$$

$$\frac{d\phi}{dt} = \pm\left[\frac{E}{2ml^2} - \frac{g}{l}\sin^2\phi\right]^{\frac{1}{2}}.$$

$$\int\frac{d\phi}{\sqrt{\frac{E}{2ml^2} - \frac{g}{l}\sin^2\phi}} = \int dt = t - t_0$$

The integral can be evaluated in terms of elliptic functions.

References

[1]